Pen And Paper Level: Easy/Intermediate

We consider gauge theories in 3+1 dimensions. Let the gauge group be G and the matter fields are in a representation R. The one loop approximation for the beta function is

$$\frac{d\alpha}{dt} = -2\beta_0$$

with $\alpha = 16\pi^2/g^2$ and g is the standard gauge coupling and t is the RG time $t = \log\mu$. We know that

$$\beta_0 = \frac{4}{3}T_2(R) - \frac{11}{3}C_2(G) \ ,$$

where T_2, C_2 are defined as usual.

1.) What are $T_2(R)$ and $C_2(G)$ for $G = SU(N)$ and N_f fermions in the fundamental+anti-fundamental representation (we will henceforth refer to this theory as 'QCD')?

2.) When do we get $\beta_0 > 0$? $\beta_0 < 0$? What happens to the coupling constant g in the infrared in each case? in the ultraviolet?

Let us now add the two-loop term to the beta function

$$\frac{d\alpha}{dt} = -2\beta_0 - \frac{2\beta_1}{\alpha} \ ,$$

and we know that

$$\beta_1 = \left(\frac{20}{3}C_2(G) + 4C_2(R)\right)T_2(R) - \frac{34}{3}(C_2(G))^2$$

Now we may have a fixed point, α^* such that $\frac{d\alpha}{dt}$ vanishes at α_*. For such a fixed point to make sense we need to assume that $\beta_0/\beta_1 < 0$.

3.) When does this condition hold for QCD? Find an expression for α_*. Sketch the behaviour of the flow for $\beta_0 > 0$ and $\beta_0 < 0$ and for $\alpha > \alpha_*$ and $\alpha < \alpha_*$.

4.) A fixed point is called a UV fixed point if the gauge coupling is relevant and it is called an infrared fixed point if it is irrelevant. Explain for which cases the above fixed point α_* is an ultraviolet fixed point and for which cases it is an infrared fixed point.

5.) What is the dimension of the operator that corresponds to changing α at the fixed point?

6.) We expect intuitively that for small enough N_f at fixed N_c the theory would confine and break chiral symmetry so it would no longer be a nontrivial conformal theory. Can you see hints that something goes wrong with the fixed point at α_* for small enough N_f?