Perturbative Field Theory in AdS and the OPE

Free field theories in AdS are dual to 'Generalized Free CFTs', which aren't quite CFTs (officially) because they do not have a stress-energy tensor. GFTs have purely Gaussian CFT correlators, which means that their $2n$-point correlators are given in terms of the 2-pt correlators by Wick contraction; for example

$$\begin{aligned}\langle\mathcal{O}(x_1)\cdots\mathcal{O}(x_4)\rangle &= \langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle\langle\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle+\langle\mathcal{O}(x_1)\mathcal{O}(x_3)\rangle\langle\mathcal{O}(x_2)\mathcal{O}(x_4)\rangle \\ &\quad +\langle\mathcal{O}(x_1)\mathcal{O}(x_4)\rangle\langle\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle \\ &= \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}}+\frac{1}{x_{13}^{2\Delta}x_{24}^{2\Delta}}+\frac{1}{x_{14}^{2\Delta}x_{23}^{2\Delta}} \end{aligned} \tag{1}$$

where we define $\mathcal{O}$ by extrapolating a free bulk field ϕ to the boundary via

$$\mathcal{O}(x)=\lim_{z\to 0} z^{-\Delta}\phi(x,z) \tag{2}$$

at the level of quantum operators. GFTs satisfy the bootstrap equations exactly.

A) GFT OPEs: In the case of GFTs, what operators are in the OPE $\mathcal{O}(x)\mathcal{O}(0)$? In the case where ϕ and $\mathcal{O}$ are scalars, compute the OPE coefficients of the first few (lowest dimension) operators in this OPE. The primary operators that occur here are often called 'double-trace' or 'double-twist' and are linear combinations of $\partial^m\mathcal{O}(0)\partial^n\mathcal{O}(0)$ with dimension $2\Delta+m+n$. You should be able to set this up as a brute force calculation in Mathematica using the fact that the correlator must have a conformal block decomposition. (If you've done this before, feel free to skip ahead.)

If we study a weakly coupled Lagrangian theory in AdS, then we will obtain boundary correlators that are well approximated by the GFT form (which is analogous to 'free propagation') plus perturbative corrections.

B) OPEs for $g\phi^3$ Theory: Now let's consider the AdS lagrangian

$$\mathcal{L}=\frac{1}{2}(\nabla\phi)^2-\frac{1}{2}m^2\phi^2-\frac{g}{6}\phi^3 \tag{3}$$

at weak coupling. At first order in g the interactions induce a non-vanishing 3-pt correlator $\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle$. Without fully evaluating any AdS Feynman diagrams, and up to an overall constant, what form does this correlator take? Thus how does the interaction affect the $\mathcal{O}(x)\mathcal{O}(0)$ OPE to leading order in g? Now think about the various Feynman diagrams that one can draw at order g^2 and g^3. Which other new operators appear in the $\mathcal{O}(x)\mathcal{O}(0)$ at these orders? On a qualitative level, how are the OPE coefficients (including those that were already non-zero in the GFT limit where $g=0$) affected by the interactions?

Next let's consider a scalar theory in AdS with a $\lambda\phi^4$ interaction. This produces a CFT correlator

$$\langle\mathcal{O}(P_1)\cdots\mathcal{O}(P_4)\rangle\propto\lambda D_\Delta^d(P_i)\equiv\frac{\lambda}{\pi^{d/2}}\int d^{d+1}X\prod_{i=1}^{4}(-2X\cdot P_i)^{-\Delta} \tag{4}$$

In general these 'D-functions' are complicated. But in the special case of $\Delta = 1$ they take a fairly simple closed form. If we place the four external points at the conventional locations $\langle \mathcal{O}(0)\mathcal{O}(z)\mathcal{O}(1)\mathcal{O}(\infty)\rangle$ then

$$D(z,\bar{z}) = \frac{1}{z-\bar{z}}\left(2\text{Li}_2(z) - 2\text{Li}_2(\bar{z}) + \log(z\bar{z})\log\frac{1-z}{1-\bar{z}}\right) \tag{5}$$

This provides a concrete, tractable model for the effect of AdS interactions on a CFT 4-pt correlator.

C) Anomalous Dimensions from ϕ^4 Theory: For convenience let's choose $\Delta = 1$ and normalize λ so that $\lambda D(z,\bar{z})$ is the full order λ correction to the 4-pt correlator. Expand $D(z,\bar{z})$ in the OPE limit and compute the perturbative shift in OPE coefficients of the operators you discovered in part A. It's easy to see that there are some terms that have a logarithmic dependence on $z\bar{z}$. These correspond to perturbative shifts in the dimensions of operators, since $z^{a+b\lambda} \sim z^a(1 + b\lambda\log z + \cdots)$. Show that only scalar primary operators recieve anomalous dimensions, and that the double-twist scalars of (zeroth order) dimension $2\Delta + 2n$ get anomalous dimensions

$$\gamma(n) \propto \frac{\lambda}{1+2n} \tag{6}$$

in $d = 2$. You may not be able to prove this formula in general, but you should be able to identify the pattern by computing the result for the first many values of n using Mathematica.

D) Dimensional Analysis and Effective Field Theory In general spacetime dimensions and at large $n \gg \Delta$, the anomalous dimension behaves as

$$\gamma(n) \sim \frac{\lambda}{n^{3-d}} \tag{7}$$

By thinking about what CFT anomalous dimensions mean in AdS, and by considering the dimension of the interaction term $\lambda\phi^4$ in $d+1$ spacetime dimensions, suggest a qualitative explanation for this large n scaling. How would you expect anomalous dimensions $\gamma(n)$ to scale at large n if in place of ϕ^4 we studied the interaction term $(\partial\phi)^4$ or ϕ^3 (for the latter, we would need to work to second order in the coupling)?

One can show that perturbative unitarity requires $|\gamma(n)| < 4$, which means that AdS effective field theories must break down when the anomalous dimensions become large. This provides an AdS/CFT analog of familiar statements about effective field theory in flat spacetime, and their relationship with unitary scattering amplitudes.