CFTs and Black Holes

In the case of AdS_3/CFT_2, we have two tools that are unavailabe in higher dimensions – the infinite dimensional Virasoro symmetry, which encodes a great deal about quantum gravity, and modular invariance on the torus, which tells us a great deal about the spectrum.

Some Basics

Feel free to skip exercises with which you are already familiar.

A) KMS Condition Show that a finite temperature CFT 2-pt function of idential real operators

$$\langle \mathcal{O}(t_L, \vec{x})\mathcal{O}(0, \vec{x})\rangle_\beta = \sum_\psi e^{-\beta E_\psi} \langle \psi | \mathcal{O}(t_L, \vec{x})\mathcal{O}(0, \vec{x}) | \psi \rangle \tag{1}$$

(where t_L is a Lorentzian time separation) must be periodic under $t_L \to t_L + i\beta$.

B) Thermal Correlators Consider the 2-pt correlator $\langle \mathcal{O}(z)\mathcal{O}(0)\rangle = \frac{1}{z^{2h}\bar{z}^{2h}}$ in a 2d CFT. Now consider the conformal mapping of the plane to the cylinder via

$$z \to e^{x + i\frac{2\pi}{\beta}t}, \quad \bar{z} \to e^{x - i\frac{2\pi}{\beta}t} \tag{2}$$

Assuming that $\mathcal{O}(z)$ is a primary operator, what is its 2-pt correlator on the cylinder? Since t is a Euclidean time, this correlation function is thermal. Note that this mapping also applies to general n-pt correlators. What is the expectation value of the CFT_2 stress-energy tensor on the cylinder (recall that the stress tensor is not a primary operator)?

C) BTZ Black Hole The (scalar) BTZ black hole metric takes the form

$$ds^2 = -(r^2 - r_+^2)dt_L^2 + \frac{dr^2}{r^2 - r_+^2} + r^2 d\phi^2 \tag{3}$$

which is locally just AdS_3 (so it's equivalent to AdS_3 with global identifications). Consider the geometry near $r = r_+$ and show that to avoid a conical singularity for Euclidean times $t = it_L$, we must impose periodicity under $t_L \to t_L + i\beta$. Determine β in terms of r_+.

What is the topology of the Euclidean boundary (where the CFT 'lives'), and how does it differ from part B? Are all boundary (CFT) correlators in the BTZ black hole background determined by conformal symmetry? Find a way to take a limit (scaling r_+) so that the answer to this last question becomes yes.

D) Free Field Correlators in a BTZ Background Compute the boundary (CFT) 2-pt correlator in the BTZ background. For this purpose it's easiest to think in Euclidean time and then use the method of images (by either summing over images in Euclidean time or in ϕ). You should find

$$\langle \mathcal{O}(t, \phi)\mathcal{O}(0, 0)\rangle_{BTZ} = \sum_{n=-\infty}^{\infty} \frac{(2\pi T)^{2\Delta} e^{-\Delta t}}{\left(4 \sin\left(\pi T_H (t + i\phi + 2\pi i n)\right) \sin\left(\pi T_H (t - i\phi - 2\pi i n)\right)\right)^{\Delta}} \tag{4}$$

where $\Delta = 2h$ and $\mathcal{O}$ is a scalar with $h = \bar{h}$. Would the method of images work in higher dimensions?

Now using the identifications $1 - z = e^{-t+i\phi}$ and $1 - \bar{z} = e^{-t-i\phi}$ expand this correlator in the OPE limit of small $z, \bar{z}$. Which operators appear in this OPE? Which Virasoro blocks are contributing? Suggest another way to compute their OPE coefficients based on Virasoro symmetry, and verify your prediction to lowest order. This will require a relation between T_H and the mass of the black hole (the dimension of the corresponding CFT operator), which you can derive.

The Quickest Derivation of the Virasoro Vacuum Block

We build representations of the Virasoro algebra by starting with a primary state $|h\rangle$ with

$$L_0|h\rangle = h|h\rangle, \quad L_{n>0}|h\rangle = 0 \tag{5}$$

and then acting on this primary with any number of Virasoro generators $L_{m<0}$. Acting with L_{-1} creates global conformal descendants (since L_{-1} is part of the global or sl(2) conformal algebra, as it is generates special conformal transformations), whereas acting with $L_{m<-1}$ creates full Virasoro descendants, which are 'gravitons' in AdS_3.

A) Null Descendants Generically, the actions of the L_{-m} are linearly independent, but there are special *degenerate states* where this is not the case. These degenerate states have null descendants. The simplest possible example is a state

$$\left(L_{-2} + xL_{-1}^2\right)|h\rangle = 0 \tag{6}$$

Find the relations between x, c, and h so that this state is primary and null. Note that h becomes non-unitary ($h < 0$) as $c \to \infty$. (This non-unitary value will be acceptable for our purposes because the Virasoro blocks are analytic in external operator dimensions.)

B) Differential Equations for Correlators By the operator state correspondence, a degenerate state $|h\rangle = \phi(0)|0\rangle$ for some local operator $\phi(z)$. Now consider the correlator

$$\mathcal{A}(z) = \langle \phi(0)\phi(z)\mathcal{O}(1)\mathcal{O}(\infty)\rangle \tag{7}$$

where $\mathcal{O}$ is an arbitrary local primary. Show that the fact that $\left(L_{-2} + xL_{-1}^2\right)$ annihilates $|h\rangle$ turns into a second order differential equation for the correlator $\mathcal{A}(z)$. This equation can be used to determine 2d Ising model correlators, but we will use it for a different purpose.

C) The Semiclassical Heavy-Light Vacuum Block Virasoro conformal blocks have a semi-classical limit when $c \to \infty$ with operator dimensions h/c fixed. This corresponds with the physics of heavy objects moving around in AdS_3 under the influence of classical gravity (recall $c = \frac{3}{2G_N R_{AdS}}$). In the case of the semiclassical heavy-light vacuum block, this means

$$\mathcal{V}(h_L, h_H, z) \approx e^{cf\left(\frac{h_L}{c}, \frac{h_H}{c}, z\right)} \tag{8}$$

Furthermore, it turns out that f has a well-defined series expansion in $\frac{h_L}{c}$, so

$$\mathcal{V}(h_L, h_H, z) \approx e^{h_L g\left(\frac{h_H}{c}, z\right)} \tag{9}$$

This relation must hold in the limit $c \to \infty$ with h_H/c and h_L fixed. In particular, it must hold in the special case when h_L is the dimension of a degenerate operator (and thus h_L depends on c, but has a finite $c \to \infty$ limit). Apply your differential equation from the previous problem to $\mathcal{V}$ in this last form, identifying h_L with the degenerate operator. Solve this equation and compute the function $g(\frac{h_H}{c}, z)$ to determine the semiclassical heavy-light Virasoro vacuum block. You should find agreement with the $n = 0$ term of the sum in equation (4).