Conformal Bootstrap - Problem Set I

The $D = 1$ Bootstrap

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• The ρ representation

Consider operators inserted on a plane, as displayed in figure 1.

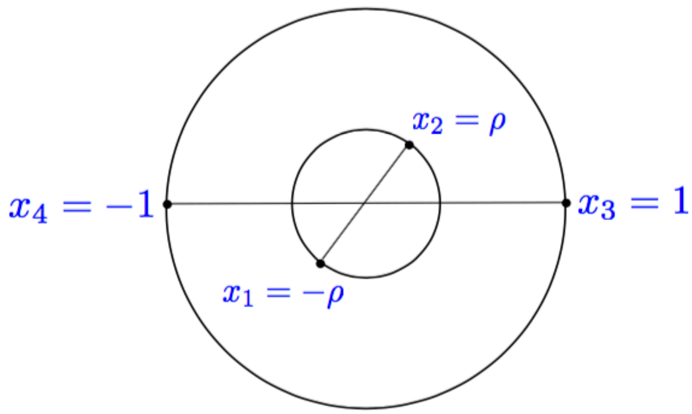

Figure 1: The ρ representation

1. Compute the cross-ratios u, v in terms of $\rho, \bar{\rho}$ and obtain the transformation between z and ρ.
2. Should conformal blocks have positive coefficients when expanded in the ρ variable? Under which conditions? Check it. Is convergence of the expansion better or worse than with z?
3. Recall the form of the conformal block in $D = 1$ for equal external operator dimensions,

$$G_\Delta(z) = z^\Delta \, {}_2F_1(\Delta, \Delta, 2\Delta, z) \tag{1}$$

   Now:

   (a) Derive the Casimir equation satisfied by the block using the conformal algebra, and prove that this expression satisfies it.

   (b) Using the Casimir equation prove that

$$G_\Delta(z) = (4\rho(z))^\Delta \, {}_2F_1\left(\Delta, \frac{1}{2}, \Delta + \frac{1}{2}, \rho(z)^2\right) \tag{2}$$

   provides an equally good representation of the $D = 1$ conformal block in terms of the ρ variable.

• Toy bootstrap and extremality

Consider the following toy crossing equation

$$0 = \frac{1}{z^{2\Delta_\phi}} - \frac{1}{(1-z)^{2\Delta_\phi}} + \sum_{\Delta>\Delta_*} c_\Delta \left(z^{\Delta-2\Delta_\phi} - (1-z)^{\Delta-2\Delta_\phi} \right). \tag{3}$$

Bounds obtained for this equation apply equally well to the real one, although typically they will not be optimal (why?).

1. Determine the c_Δ for the generalized free fermion (GFF) four point function:

$$\langle \phi(x_1) \dots \phi(x_4) \rangle = \frac{1}{x_{13}^{2\Delta_\phi} x_{24}^{2\Delta_\phi}} \left(-1 + z^{-2\Delta_\phi} + (1-z)^{-2\Delta_\phi} \right) \tag{4}$$

2. Find analytically the optimal upper bound on $\Delta^*(\Delta_\phi)$ by considering functionals in the basis $(\partial_z, \partial_z^3)|_{z=1/2}$.

3. Do the same but now up to ∂_z^5 derivatives. Any surprises?

4. Let us be ambitious and add one more derivative (i.e. up to ∂_z^7). Focus on a specific value of Δ_ϕ for simplicity, and don't bother too much with exact results. The goal is to find the optimal bound on Δ^* and extremal functional. (Hint: use the fact that at extremality there is a unique solution to crossing to find the bound). What are the approximate dimensions and "OPE" coefficients? Are we close to the GFF solution?

5. (Longish) Now repeat the whole analysis for the real crossing equation, obtaining exact results if possible. How do the resulting dimensions and OPE coefficients compare with the ones for the GFF (you should compute the first few terms in the conformal block decomposition)? How does the comparison do for different values of Δ_ϕ? Compare with the speed of convergence of the crossing equation for different values of Δ_ϕ.