

Conformal Bootstrap - Problem Set II

The $D > 1$ Bootstrap

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The goal of this exercise is to get you to obtain your first kink, or at least your very first non-trivial bounds in higher dimensional CFTs. We will focus on $D = 2$, since in this case the conformal blocks are known explicitly and there is a nice kink at the position of the 2d Ising model. All computations should be done in **Mathematica**!

The $D = 2$ conformal blocks are given by

$$\begin{aligned} G_{\Delta,L}(z, \bar{z}) &= \frac{1}{2} \left(k_{\Delta-L}(z) k_{\Delta+L}(\bar{z}) + k_{\Delta-L}(\bar{z}) k_{\Delta+L}(z) \right) \\ k_{\beta}(z) &:= z^{\frac{\beta}{2}} {}_2F_1(\beta/2, \beta/2, \beta, z). \end{aligned} \quad (1)$$

Now, recall the crossing sum-rule:

$$F_{0,0}^{\Delta\phi}(z, \bar{z}) + \sum_{\Delta,L} \lambda_{\Delta,L}^2 F_{\Delta,L}^{\Delta\phi}(z, \bar{z}) = 0 \quad (2)$$

with

$$F_{\Delta,L}^{\Delta\phi}(z, \bar{z}) = \frac{G_{\Delta,L}(z, \bar{z})}{z^{\Delta\phi} \bar{z}^{\Delta\phi}} - \frac{G_{\Delta,L}(1-z, 1-\bar{z})}{(1-z)^{\Delta\phi} (1-\bar{z})^{\Delta\phi}}. \quad (3)$$

We will choose a particular value of Δ_ϕ – let's start with $1/8$, the dimension of the σ operator in the 2d Ising model – and derive an upper bound on the dimension of the lowest scalar operator in the OPE $\phi \times \phi$.

The first step is to turn the continuously infinite set of constraints contained in the crossing sum rule into a finite, discrete set. The second step is to choose a grid in the space of quantum numbers; we will not be able to allow all Δ, L on the computer!

1. Define a grid Z in z, \bar{z} space. Define a grid D in Δ, L space. For each pair (Δ, L) in D evaluate numerically the $F_{\Delta,L}^{\Delta\phi}$ on the grid Z to obtain a set of vectors $\vec{v}_{\Delta,L}$.

Remarks:

- There is a lot of freedom in the choice of grids. The finer the better, but computations will become slower. You should aim for vectors of size $\lesssim 10$. A Z grid clustered around $z = \bar{z} = 1/2$, with $\bar{z} = z^*$ also works reasonably well. Remember the symmetry of the block under $z \leftrightarrow \bar{z}$ and antisymmetry of $F_{\Delta,L}^{\Delta\phi}$ under $z, \bar{z} \rightarrow 1-z, 1-\bar{z}$ to avoid repeating points.

- The grid D containing spins from 0 to 8 or 10 at the most (in steps of 2; only even spins appear in the OPE of two identical scalars). The values of Δ should be those consistent with unitarity ($\Delta \geq L$ in $D = 2$), up to some Δ_{Max} of order a couple of dozens, with a rough spacing of order 10^{-1} .
- It is sufficient to evaluate everything with machine precision. It is better to split the computation into two steps, one where we compute the blocks on the Z grid and another where one includes the factors involving Δ_ϕ . This is useful because the second step is much faster than the first, and we'll eventually be interested in repeating the computation for many different Δ_ϕ .

To derive a bound we must show that if we exclude all scalar operators below some dimension Δ_{gap} then it is possible to construct a linear functional $\vec{\omega}$ such that

$$\begin{aligned} \vec{\omega} \cdot \vec{v}_{\Delta,L} &\geq 0, & \begin{cases} L = 0 : \Delta \geq \Delta_{\text{gap}} \\ L \geq 2 : \Delta \geq L \end{cases} \\ \vec{\omega} \cdot \vec{v}_{0,0} &> 0. \end{aligned} \tag{4}$$

Notice the strict inequality sign. We want to find such functionals using `Mathematica`, and this can be achieved by using the function `LinearProgramming`. This function requires some objective function that we are trying to minimize. Since we are merely asking for one functional satisfying the constraints, we can choose the objective to be trivial, such that our linear program is

$$\min \vec{\omega} \cdot \vec{0} \quad \text{subject to constraints above.} \tag{5}$$

2. Implement the linear program using the `LinearProgramming` function. How low can you make Δ_{gap} ? That by definition represents the optimal bound Δ^* , for the given Z grid.

If `Mathematica` gives you a non-trivial functional for some Δ_{gap} , congratulations! That is an upper bound on the dimension of the leading scalar operator. Or is it? Well, actually this is too quick. In order to prove that such a bound is correct, we would need to show that refining the grid D , adding more spins, etc, does not affect the result *much* – so I encourage you to experiment with this a bit. See however the first remark below.

Remarks:

- We would not like the precise bound to be constrained by the coarseness of our grid. A way around this is to first remove from our matrix M all scalar vectors below the gap Δ^* ; and then include an extra vector with Δ equal precisely to Δ^* . In this way we can obtain a very precise value of Δ^* by *bisection*: we try one value, if no functional exists we increase the gap, if a functional exists we lower it, alternating until we've narrowed down Δ^* to some desired accuracy.
- The precise value one picks on the RHS of the constraint $\vec{\omega} \cdot \vec{v}_{0,0} \geq \epsilon$ acts as an effective normalization of the functional. A problem with this is that at extremality, i.e. when Δ^* is optimal, the functional wants to annihilate $\vec{v}_{0,0}$. A better way to proceed is to relax the constraint to $\vec{\omega} \cdot \vec{v}_{0,0} \geq 0$ and to add a new constraint, such as

$$\vec{\omega} \cdot \partial_\Delta \vec{v}_{\Delta,0}|_{\Delta=\Delta^*} \geq 1 \tag{6}$$

which guarantees the functional starts out positive at the gap.

- We have the freedom to rescale each individual component of all vectors. The numerics will be more stable say if we divide all vectors by a particular one, say the vector $\partial_{\Delta} \vec{v}_{\Delta,0}|_{\Delta=\Delta^*}$ given above.
- Plotting the functionals, i.e. plotting the functions

$$\omega_L(\Delta) := \vec{\omega} \cdot \vec{v}_{\Delta,L}, \quad (7)$$

one should see functions which are positive semidefinite in the appropriate regions. For reasonable grids (where reasonable is hard to quantify), the only mistake induced by discretization is that the functional may dip down and have a zero at some point on the grid D . In between the grid points the functional will then dip below zero. This means our functionals are strictly speaking not well defined, and this leads to the bound being stronger than it should be. The way around this is to refine the grid by adding extra points near the zeros.

I would recommend implementing the first three tricks; the last is more ambitious and might not be necessary. Either way, your task now is to:

3. Repeat the analysis for several values of Δ_{ϕ} and plot $\Delta^*(\Delta_{\phi})$. What does the bound look like? What about the bound at $\Delta_{\phi} = 1/8$, is it close to 1 as for the 2d Ising model? What does the functional look like on either side of the kink?

To see a kink will require some persistence, but the result will be worth it!

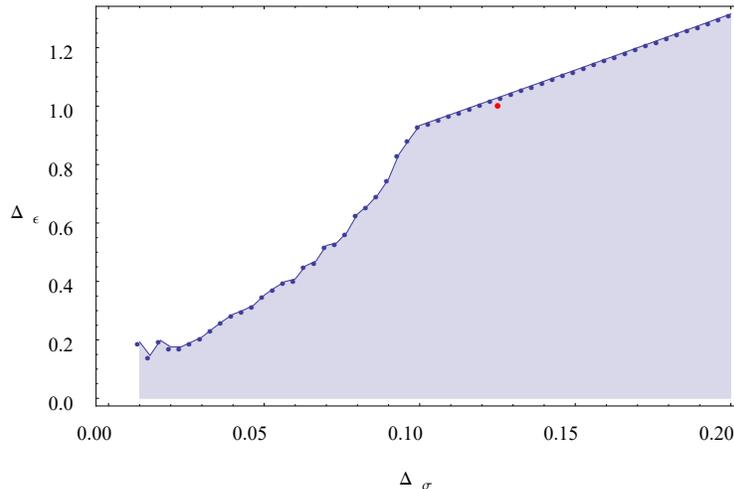


Figure 1: A bound on the scaling dimension of the leading scalar ε appearing in the OPE $\sigma \times \sigma$. Obtained by following this problem set, using 10 components. In red the 2d Ising model at $(1/8, 1)$. As we increase the number of components, the kink moves towards that point.