1. Show that the bubble diagram

\[ \begin{array}{c}
\text{\textbullet} \\
| \\
| \\
\text{\textbullet} \\
\end{array} \]

has a square root cut, i.e. behaves like \( \sqrt{s - 4m^2} \) near \( s = 4m^2 \).

2. Consider a theory with \( \mathcal{L} = \frac{1}{4} g \phi^2 \chi^2 \) in \( d = 4 \). Using your previous answer, give the \( \phi - \phi \) elastic scattering amplitude to second order in \( g \). Use the partial wave decomposition to verify that the unitarity condition is satisfied up to order \( g^2 \).

3. Consider the elastic scattering of two real scalar particles of mass \( m^2 \) in \( d = 4 \). Suppose the amplitude is real-analytic and saturates unitarity for a finite range of \( s \) near threshold. Prove that the amplitude must have a two-sheeted branch cut starting at \( s = 4m^2 \).

4. Compute the location of the anomalous threshold singularity in the real \((s,t)\) plane for the Landau diagram:

\[ \begin{array}{c}
\text{\textbullet} \\
| \\
| \\
\text{\textbullet} \\
\end{array} \]

You may suppose that all momenta are Euclidean and planar. Hint: make a simple ansatz for the Euclidean momenta like \( p_1 = m \cos(\phi), \sin(\phi) \), etc.

5. Consider scattering amplitudes in two spacetime dimensions, again for real identical scalar particles. Using the rapidity variables \( \theta_i \), defined such that \( p_i = m \cosh(\theta_i), \sinh(\theta_i) \), show that for on-shell momenta

\[
\delta^{(2)}(p_1 + p_2 - p_3 - p_4) \propto \delta(p_1 - p_4)\delta(p_2 - p_3) + (2 \leftrightarrow 3)
\]

and find the exact proportionality constant. Show that we can therefore write

\[
(p_3, p_4|S|p_1, p_2) = (2\pi)^2 E_1 E_2 (\delta(p_1 - p_4)\delta(p_2 - p_3) + (1 \leftrightarrow 2)) S(s)
\]

with \( s = (p_1 + p_2)^2 \) the only remaining Mandelstam invariant. Show that crossing symmetry implies that \( S(4m^2 - s) = S(s) \). What is the constraint of unitarity? Give \( S(s) \) for a simple contact interaction corresponding to \( \mathcal{L}_{int} = \lambda \phi^4/(4!) \).
6. Consider the Virasoro-Shapiro amplitude (in units such that $\alpha' = 1$)

$$T_{VS}(s, t) = \left( \frac{tu}{s} + \frac{su}{t} + \frac{st}{u} \right) \frac{\Gamma(1 - s/4)\Gamma(1 - t/4)\Gamma(1 - u/4)}{\Gamma(1 + s/4)\Gamma(1 + t/4)\Gamma(1 + u/4)}$$

(3)

and the partial wave expansion

$$T_{VS}(s, t) = \sum_{\ell=0}^{\infty} f_\ell(s)C_\ell(\cos(\theta))$$

(4)

where now $\cos(\theta) = 1 + 2t/s$ and the $C_\ell(z) = c_\ell z^\ell + \ldots$ are polynomials of degree $\ell$; you can think of these as ten-dimensional versions of the Legendre polynomials. Notice also that the external particles are massless so $s + t + u = 0$.

(a) Show that the residues of the scattering amplitude at $s = 4n$ are polynomials in $\cos(\theta)$ of degree $2n + 2$. For which values of $n$ does $f_\ell(s)$ then have a pole?

(b) Consider only the poles in $f_\ell(s)$ at $s = 2\ell - 4$, which form the leading Regge trajectory. Use this trajectory together with crossing symmetry to show that

$$T_{VS}(s, t) \sim s^{2+t/2}$$

(5)

for large $s$ and fixed $t$. Check that this is the correct result by explicitly taking the limit of $T_{VS}(s, t)$. Can you match the exact coefficient as well?