The S-matrix Bootstrap

Problem sheet 1

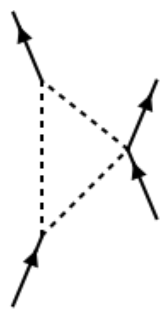
1. Show that the bubble diagram

   has a square root cut, i.e. behaves like $\sqrt{s-4m_\chi^2}$ near $s=4m_\chi^2$.

2. Consider a theory with $\mathcal{L}=\frac{1}{4}g\phi^2\chi^2$ in $d=4$. Using your previous answer, give the ϕ-ϕ elastic scattering amplitude to second order in g. Use the partial wave decomposition to verify that the unitarity condition is satisfied up to order g^2.

3. Consider the elastic scattering of two real scalar particles of mass m^2 in $d=4$. Suppose the amplitude is real-analytic and saturates unitarity for a finite range of s near threshold. Prove that the amplitude must have a two-sheeted branch cut starting at $s=4m^2$.

4. Compute the location of the anomalous threshold singularity in the real (s,t) plane for the Landau diagram:



   You may suppose that all momenta are Euclidean and planar. Hint: make a simple ansatz for the Euclidean momenta like $p_1=m(\cos(\phi),\sin(\phi))$, etc.

5. Consider scattering amplitudes in *two* spacetime dimensions, again for real identical scalar particles. Using the rapidity variables θ_i, defined such that $p_i=m(\cosh(\theta_i),\sinh(\theta_i))$, show that for on-shell momenta

$$\delta^{(2)}(p_1+p_2-p_3-p_4)\propto\delta(\mathbf{p}_1-\mathbf{p}_4)\delta(\mathbf{p}_2-\mathbf{p}_3)+(2\leftrightarrow 3) \tag{1}$$

   and find the exact proportionality constant. Show that we can therefore write

$$\langle\mathbf{p}_3,\mathbf{p}_4|S|\mathbf{p}_1,\mathbf{p}_2\rangle=(2\pi)^2 4E_1E_2\left(\delta(\mathbf{p}_1-\mathbf{p}_4)\delta(\mathbf{p}_2-\mathbf{p}_3)+(1\leftrightarrow 2)\right)S(s) \tag{2}$$

   with $s=(p_1+p_2)^2$ the only remaining Mandelstam invariant. Show that crossing symmetry implies that $S(4m^2-s)=S(s)$. What is the constraint of unitarity? Give $S(s)$ for a simple contact interaction corresponding to $\mathcal{L}_{int}=\lambda\phi^4/(4!)$.

6. Consider the Virasoro-Shapiro amplitude (in units such that $\alpha' = 1$)

$$T_{VS}(s,t) = \left(\frac{tu}{s} + \frac{su}{t} + \frac{st}{u}\right) \frac{\Gamma(1-s/4)\Gamma(1-t/4)\Gamma(1-u/4)}{\Gamma(1+s/4)\Gamma(1+t/4)\Gamma(1+u/4)} \tag{3}$$

and the partial wave expansion

$$T_{VS}(s,t) = \sum_{\ell=0}^{\infty} f_\ell(s) C_\ell(\cos(\theta)) \tag{4}$$

where now $\cos(\theta) = 1 + 2t/s$ and the $C_\ell(z) = c_\ell z^\ell + \dots$ are polynomials of degree ℓ; you can think of these as ten-dimensional versions of the Legendre polynomials. Notice also that the external particles are massless so $s + t + u = 0$.

(a) Show that the residues of the scattering amplitude at $s = 4n$ are polynomials in $\cos(\theta)$ of degree $2n + 2$. For which values of n does $f_\ell(s)$ then have a pole?

(b) Consider *only* the poles in $f_\ell(s)$ at $s = 2\ell - 4$, which form the *leading* Regge trajectory. Use this trajectory together with crossing symmetry to show that

$$T_{VS}(s,t) \sim s^{2+t/2} \tag{5}$$

for large s and fixed t. Check that this is the correct result by explicitly taking the limit of $T_{VS}(s,t)$. Can you match the exact coefficient as well?