CFT Basics - Problem I

Conformal Symmetry and Correlators

Christopher Beem

Recall from the first lecture that a special conformal transformation acts on coordinates in $\mathbb{R}^d$ as

$$x'^{\mu} = e^{iK\cdot b} x^{\mu} = \frac{x^{\mu} + b^{\mu} x^2}{1 + 2(b\cdot x) + (b\cdot b)(x\cdot x)} \ ,$$

while an inversion acts according to

$$x'^{\mu} = I(x)^{\mu} = \frac{x^{\mu}}{x\cdot x} \ .$$

(i) Show that this transformation can be produced by conjugating a finite translation with inversions,

$$e^{iK\cdot b} = I \circ e^{iP\cdot b} \circ I \ .$$

(ii) Show that the distance between two points in $\mathbb{R}^d$ transforms under a special conformal transformation according to

$$|x_1' - x_2'| = \frac{|x_1 - x_2|}{\gamma_1^{1/2}\gamma_2^{1/2}} \ ,$$

where $\gamma_i := 1 + 2(b\cdot x_i) + (b\cdot b)(x_i\cdot x_i)$.

(iii) Now show that under an arbitrary conformal transformation, the distance between two points transforms according to

$$|x_1' - x_2'| = \Omega(x_1)^{1/2}\Omega(x_2)^{1/2}|x_1 - x_2| \ ,$$

where $\Omega(x)$ is the local scale factor for the conformal transformation at x,

$$\delta_{\mu\nu}\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}} = \Omega^2(x)\delta_{\alpha\beta} \ .$$

Now consider a correlation function of conformal primary scalar operators. Such a correlator will obey the conformal Ward identities

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \Omega_1^{\Delta_1}\cdots\Omega_n^{\Delta_n}\langle \mathcal{O}_1(x_1')\cdots\mathcal{O}_n(x_n')\rangle \ ,$$

where the Δ_i are the scaling dimensions of the operators and $\Omega_i := \Omega(x_i)$.

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(*iv*) Prove that the two-point function for primary scalar operators with dimensions Δ_1 and Δ_2 is

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{\kappa_{12}\delta_{\Delta_1,\Delta_2}}{x_{12}^{2\Delta_1}} \ ,$$

where we define $x_{ij} = |x_i - x_j|$.

(*v*) Prove that the three-point function of primary scalar operators with dimensions Δ_i is

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{\lambda_{123}}{x_{12}^{\Delta_{123}}x_{23}^{\Delta_{231}}x_{31}^{\Delta_{312}}} \ ,$$

where

$$\Delta_{ijk} := \Delta_i + \Delta_j - \Delta_k \ .$$

(*vi*) Prove that the most general form allowed by the conformal Ward identities for the four-point function of identical scalar primary operators of scaling dimension Δ is

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}}F(u,v) \ ,$$

where u and v are *conformal cross ratios* that you should define. These are combinations of the coordinates x_i that are invariant under arbitrary conformal transformations, so the function F can in principle be arbitrary.

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CFT Basics - Problem II

The Operator Product Expansion

Christopher Beem

Consider the operator product expansion of scalar primaries,

$$\mathcal{O}_1(x)\mathcal{O}_2(0) \sim \sum_k C_{12k,a_1\cdots a_\ell}(x,P)\mathcal{O}_k^{(a_1\cdots a_\ell)}(0) \ ,$$

where k runs over all conformal primaries and Δ_k and ℓ refer to the scaling dimension and spins of those operators.

(i) Show that scale invariance ($i.e.$, acting on both sides with the dilatation operator D) implies that the differential operator C must obey

$$C_{12k,a_1\cdots a_\ell} = \frac{f_{12k}}{|x|^{\Delta_{12k}+\ell}} F_{12k,a_1\cdots a_\ell}(x,P) \ ,$$

where $F_{12k,a_1\cdots a_\ell}$ obeys the homogeneity condition

$$F_{12k,a_1\cdots a_\ell}(\lambda x, \lambda^{-1}P) = \lambda^\ell F_{12k,a_1\cdots a_\ell}(x,P) \ .$$

(ii) Specializing to the case where $\mathcal{O}_k$ is a scalar operator, we have an expansion in derivatives,

$$F_{12k}(x,P) = 1 + \alpha\, x^\mu P_\mu + \beta\,(x\cdot x)(P\cdot P) + \gamma\, x^\mu x^\nu P_\mu P_\nu + \dots \ .$$

Further specializing to the case where $\mathcal{O}_1$ and $\mathcal{O}_2$ both have scaling dimension Δ_ϕ, use special conformal invariance to determine the numerical coefficients α, β, and γ.

(iii) Derive the same result by applying the OPE inside the correlation function

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(0)\mathcal{O}_k(w)\rangle = \frac{f_{12k}}{|x|^{\Delta_{12k}}|w|^{\Delta_{2k1}}|x-w|^{\Delta_{k12}}} \ . \qquad (1)$$

and evaluating both sides in a series expansion.

(iv) Now specializing to the case of *one dimension* ($d=1$, you may assume that $x<0<w$), show that the differential operator $F_{12k}(x,\partial)$ obeys

$$F_{12k}(x,\partial_y)\left(\frac{w}{w-y}\right)^{2\Delta_k}\Bigg|_{y=0} = \left(\frac{w}{w-x}\right)^{\Delta_{k12}}$$

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(v) Using Mathematica, compute the coefficients in the expansion

$$F_{12k}(x,\partial) = \sum_{n=0}^{\infty} c_n x^n \partial^n \ .$$

for $n \leqslant 20$. See if you can guess an analytic formula for the c_n. Can you prove the formula?

(vi) [**HARD**] Re-do parts (iv) and (v) for the case of two-dimensional CFTs. In this case the relevant differential equation is

$$F_{12k}(x,\partial_y)\left(\frac{w}{w-y}\right)^{2\Delta_k}\Bigg|_{y=0} = \left(\frac{w}{w-x}\right)^{\Delta_{k12}}$$

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CFT Basics - Problem III

Exercises from Lecture

Christopher Beem

This is a collection of small calculations mentioned during the lectures that were left as exercises.

(*i*) Derive the differential form of the Ward identity for conservation of the stress tensor,

$$\langle \partial_\mu T^{\mu\nu}(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = -\sum_{i=1}^{n}\delta(x-x_i)\frac{\partial}{\partial(x_i)_\nu}\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle\ .$$

Recall that this follows from diffeomorphism invariance.

(*ii*) Derive the corresponding Ward identity for rotations, *i.e.*, for conservation of $\epsilon_\nu(x)T^{\mu\nu}(x)$ where ϵ is the Killing vector that generates rotations.

(*iii*) In two dimensions there are infinitely many solutions to the conformal Killing equation

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = c(x)\delta_{\mu\nu}.$$

Find them and compute their commutators (this Lie algebra is the *Witt algebra*).

(*iv*) [**painful**] Check that the algebra of conformal Killing vectors in general dimension is $\mathfrak{so}(d+1,1)$.

(*v*) Prove that in a unitary CFT, every local operator can be written as a sum of primaries and descendants.

(*vi*) Recall the definition of Hermitian conjugation for operators on the Euclidean plane acting on the sphere Hilbert space

$$(O^{a_1\cdots a_\ell}(x))^\dagger = |x|^{-2\Delta}I^{a_1}_{b_1}\cdots I^{a_\ell}_{b_\ell}(\mathcal{O}^\dagger)^{b_1\cdots b_\ell}\left(\frac{x}{|x|^2}\right)\ .$$

Deduce that the conformal charges acting on this Hilbert space obey

$$(Q_\epsilon)^\dagger = Q_{I\epsilon I}\ .$$

(*vii*) Prove the unitarity bound reported in the lectures for scalar operators,

$$\Delta(\Delta - \frac{d-2}{2}) \geqslant 0\ ,$$

by computing the norm of the state $|\psi\rangle = P_\mu P^\mu|\mathcal{O}\rangle$.

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(*viii*) [**harder**] Prove the bound reported in the lectures for traceless symmetric tensors of spin ℓ,

$$\Delta - (l + d - 2) \geqslant 0 \ ,$$

by considering states of the form $P_\mu |\Omega^{\mu\mu_2\cdots\mu_\ell}\rangle$.

(*viii*) Determine whether a conserved, even-spin symmetric traceless tensor can have a nonzero three-point function with two identical scalar operators.

(*ix*) Derive the general form for the four-point function of *non-identical* scalar operators.

(*x*) Write the conformal Casimir operator, $C = \frac{1}{2} L^{AB} L_{AB}$, in terms of the charges $M_{\mu\nu}$, P_μ, K_μ, D and check that you have the coefficients correct by verifying that it commutes with all of the charges.

(*xi*) [**hard**] Compute the differential operator that arises from inserting the conformal Casimir in the four-point function of identical scalar operators. This was derived in the paper *Conformal Partial Waves and the Operator Product Expansion* by Dolan and Osborn.