***Bootstrap Problem Solving***
**Type: Pencil and paper**
**Difficulty: Starts easy, gets hard**

## 2d Blocks, Causality, and Chaos

A 2d CFT can be organized under the $Virasoro \times Virasoro$ algebra, or under the global $SL(2) \times SL(2)$ subalgebra. Each Virasoro rep breaks up into an infinite sum of $SL(2)$ reps. A "quasiprimary" state in 2d CFT is primary under $SL(2)$:

$$L_0|h,\bar{h}\rangle = h|h,\bar{h}\rangle, \qquad \overline{L}_0|h,\bar{h}\rangle = \bar{h}|h,\bar{h}\rangle, \qquad L_1|h,\bar{h}\rangle = \overline{L}_1|h,\bar{h}\rangle = 0 \tag{0.1}$$

"Global" conformal blocks sum up the SL(2) descendants of a quasiprimary. For external operators identical in pairs, the global block is defined as

$$\mathcal{F} = F[h_1, h_2, h, z]F[\bar{h}_1, \bar{h}_2, \bar{h}, \bar{z}] \tag{0.2}$$

where the holomorphic (or "left-moving") SL(2) block is

$$F[h_1, h_2, h, z] \equiv \frac{1}{\mathcal{N}} \sum_{n \geq 0} \frac{\langle O_3(\infty)O_3(1)(L_{-1})^n|h,\bar{h}\rangle\langle h,\bar{h}|(L_1)^n O_1(z,\bar{z})O_1(0)\rangle}{\langle h,\bar{h}|(L_1)^n(L_{-1})^n|h,\bar{h}\rangle} \tag{0.3}$$

and similarly for the anti-holomorphic term. The normalization $\mathcal{N}$ is picked so that $F = z^h(1 + \cdots)$.

**1.** Derive the formula for the global block:

$$F[h_1, h_2, h, z] = z^h {}_2F_1(h, h, 2h, z) \tag{0.4}$$

Hint: A useful intermediate step is $\langle h,\bar{h}|(L_1)^n O_1(z,\bar{z})O_1(0)\rangle = \left(\frac{c_{h11}}{z^{2h_1}\bar{z}^{2\bar{h}_1}}\right)(h)_n z^{h+n}$ where $c_{ijk}$ is the OPE coefficient and $(\cdot)_n$ is the Pochhammer symbol.

Bonus: You also just derived the lightcone (or "colinear") conformal block in arbitrary dimensions.

**2.** In a generic CFT, the full Virasoro block $\mathcal{F}_{Vir}$ can be decomposed into an infinite sum of $SL(2)$ blocks of increasingly high quasiprimary weight $h$.

Let's consider the Virasoro vacuum block, which includes all Virasoro descendants of $|0\rangle$. Show that the state $L_{-2}|0\rangle$ is quasiprimary. Then, use the Virasoro algebra to argue that in the limit $c \to \infty$ with all the conformal weights held fixed, this is the only important quasiprimary in the Virasoro vacuum block. That is,

$$F_{Vir}[h_1, h_2, 0, z] = 1 + c_T F[h_1, h_2, 2, z] + O(1/c^2) \tag{0.5}$$

Compute $c_T$.

**3.** Consider a scalar 4-pt function at large $c$ (with conformal weights $h_1, h_2 \ll c$). Argue that in the lightcone limit, i.e., $\bar{z} \to 0$ with $z$ held fixed, the correlator is given by

$$\langle O_1(0) O_1(z, \bar{z}) O_2(1) O_2(\infty) \rangle \approx (z\bar{z})^{-2h_1} \left(1 + \frac{12 h_1 h_2}{c} \frac{1}{z} (-2z + (z-2)\log(1-z))\right) \tag{0.6}$$

(Assume the only conserved currents in the theory are those coming from Virasoro.) Note that despite appearances, the correction term is perfectly regular at $z = 0$.

**4.** The expression (0.6) has a branch cut at $z = 1$. Take $z$ through the branch cut (from above) and then take $z \to 0$; in this limit you should find

$$\langle O_1(0) O_1(z, \bar{z}) O_2(1) O_2(\infty) \rangle \approx (z\bar{z})^{-2h_1} \left(1 + \frac{\lambda_T}{z} + \cdots\right) \tag{0.7}$$

Find $\lambda_T$.

This regime of the correlator, after passing through the branch cut, is often called the 'second sheet' and is the regime relevant to causality constraints and the chaos bound. Note that we started with a function regular at $z \sim 0$, took it around the cut, and ended up with a function that blows up at $z \sim 0$! This blow-up encodes a lot of interesting physics: the growth is related to chaos at finite temperature, the positivity of Im $\lambda_T$ is related to causality, and the fact that it is suppressed by a $1/c$ factor is related to the fact that holographic CFTs have scrambling time $t_* \sim \log c$. We will explore a simple version of this in the next few parts (based partly on [Roberts and Stanford '14] and [Perlmutter '16]).

Consider the (Lorentzian signature) correlation function

$$G(u, v) \equiv \langle O_1(u, v) O_2(1) O_2(-1) O_1(-u, -v) \rangle \tag{0.8}$$

where $u, v = t \pm y$ are lightcone coordinates in 2d Minkowski space, in the regime

$$u < 0 < v, \quad |u| \ll \frac{1}{v} \ll 1 \ . \tag{0.9}$$

**5.** Compute $G(u, v)$ using (0.7).

**6.** Reinterpret your answer to part (6) in terms of a finite-temperature field theory by changing from Minkowski coordinates to Rindler coordinates. You should find that the $1/z$ term becomes a term that grows exponentially in (Rindler) time:

$$G \sim 1 + \times e^{\lambda_L t_{Rindler}} \tag{0.10}$$

(dropping time-independent factors). This exponential growth is the signature of chaos. Find the Lyapunov exponent $\lambda_L$. The 'bound on chaos' derived by Maldacena, Shenker, and Stanford is, in this context, the bound $\lambda_L \leq 2\pi T$ where $T$ is the temperature. Is the bound obeyed in your case? How does the correction term depend on spatial distance? (This dependence encodes what is called the 'butterfly velocity.')

**7.** Why, in part (5), did we use the 2nd sheet correlator instead of the ordinary one?

Hint: The correlator (0.8) is time ordered. One way to ensure you are computing the time-ordered correlator and not some other ordering is use the $i\epsilon$ prescription

$$\langle O_1(t_1, x_1) O_2(t_2, x_2) O_3(t_3, x_3) \cdots \rangle = \lim_{\epsilon \to 0} \langle O_1(t_1, x_1) O_2(t_2 + i\epsilon, x_2) O_3(t_3 + 2i\epsilon, x_3) \cdots \rangle \tag{0.11}$$

Now with $\epsilon$ small but finite, as you move $(u, v)$ from a Euclidean configuration to the desired Lorentzian configuration, you should check that $z$ goes through the branch cut. See section 3 of [1509.00014] for a review of Lorentzian continuations from the Euclidean.