Bootstrap Problem Solving
Type: Pencil and Paper + Mathematica + SDPB
Difficulty: Starts easy, gets hard

2d toy modular bootstrap

In this problem we will do a simple version of numerical bootstrap to constrain the spectrum and gap of a 2d CFT.

The partition function of a 2d CFT at inverse temperature β (and zero angular potential) is

$$Z(\beta) = \sum_{states} e^{-\beta E} \tag{0.1}$$

where E is the energy (of the state on the unit circle). The state operator correspondence says that for every primary operator of weights $(L_0, \overline{L}_0) = (h, \bar{h})$, there is a state on the unit circle with energy

$$E = h + \bar{h} - \frac{c}{12} \tag{0.2}$$

where c is the central charge. (The shift here is the Casimir energy.) The crossing equation, in this context, is the statement of modular invariance:

$$Z(\beta) = Z(\frac{4\pi^2}{\beta}) \ . \tag{0.3}$$

The self-dual temperature, $\beta = 2\pi$, plays the same role as the crossing-symmetric point $z = \bar{z} = \frac{1}{2}$ in the correlator bootstrap.

(Virasoro symmetry actually organizes the partition function into Virasoro characters, but for the purposes of this problem we will completely ignore Virasoro and just think of the spectrum as a set of independent states.)

1. Define the linear functional α, acting on a function $f(\beta)$, by

$$\alpha \circ f = \sum_{n=0}^{N} a_n (\beta \partial_\beta)^n f(\beta)|_{\beta=2\pi} \tag{0.4}$$

where a_n are some arbitrary parameters and N is the maximum number of derivatives that you choose to keep. (I found $N \sim 26$ a good number to start with for numerical experimentation in this problem.) Act on the crossing equation with α to conclude that

$$\sum_{states} \alpha(E) = 0, \quad \text{where} \quad \alpha(E) \equiv \sum_{n\ odd} a_n T_n(-2\pi E) \tag{0.5}$$

Here $T_n(x)$ is called a Touchard polynomial. You can look up its properties on Wikipedia.

Since $\sum \alpha(E) = 0$, this means we can rule out theories with certain spectra by showing that $\alpha(E) \geq 0$ for all allowed E.

2. Write a mathematica notebook, interfacing with SDPB, to implement the following optimization problem (within the framework above, ie ignoring Virasoro symmetry):

Consider a 2d CFT with central charge c, a vacuum state at $E = -c/12$, and no other states below the 'gap' energy E_{gap}. Subject to the constraint

$$\alpha(E) \geq 0 \quad \text{for} \quad E \geq E_{gap} \ , \tag{0.6}$$

find the functional α that maximizes the vacuum contribution $\alpha(-\frac{c}{12})$.

Denote this maximum $\alpha^* = \alpha_{optimal}(-c/12)$.

Argue that if you find $\alpha^* > 0$, then the theory is ruled out.

3. Now set $c = 12$. Run your program with $E_{gap} = \frac{1}{2}$. You should find that the theory is not ruled out – that is, $\alpha^* < 0$. For the optimal functional, plot $\alpha(E)$ and confirm that it obeys the constraint (0.6).

4. Still with $c = 12$, show that $E_{gap} = 2$ is impossible.

5. Still with $c = 12$, gradually increase E_{gap} from .5 toward 1. As you do so, plot the extremal functional $\alpha(E)$. What happens to α^* as you increase the gap? What happens to the minima of the extremal functional?

6. Still with $c = 12$, show that the maximal allowed gap is $E_{gap} \sim 1$. Close to this gap, plot the extremal functional. It should have double zeros at positive integers. This indicates that you have discovered a solution to crossing with $E_{gap} = 1$ and a spectrum of energies

$$E = -1, 0, 1, 2, 3, 4, \dots \tag{0.7}$$

[Caveat: You cannot see the state at $E = 0$ by this method because it drops out of the crossing equation.]

7. In the theory of modular forms, there is a famous function called the J-function (implemented in Mathematica as KleinInvariantJ) which obeys the identity

$$J(\tau) = J(-1/\tau) \tag{0.8}$$

and has an expansion in $q \equiv e^{2\pi i \tau}$ near $\tau = i\infty$ with only integer powers:

$$1728 J(\tau) = q^{-1} + 744 + 196884 q + 21493760 q^2 + \cdots \tag{0.9}$$

Use the J-function to construct a function $Z(\beta)$ that obeys crossing (0.3) and has the same spectrum that you discovered in part (6).

8. Now set $c = 24$, and use your program to find constraints on E_{gap}. (You should be able to find some constraints, but the optimal $N \to \infty$ constraints in this case are, to my knowledge, an unsolved problem.)