Problems on Space of CFTs

Problem 1.

Problem Style: **PEN & PAPER**

Difficulty: **MEDIUM**

$O(N)$ model with cubic anisotropy.

Consider an XY or a Heisenberg magnet whose magnetic ions are arranged in a cubic lattice. In this case, the interactions with the lattice break the $O(N)$ rotational group acting on the spin vectors $\vec{\phi} = (\phi_1, \ldots, \phi_N)$. Thus, additional terms appear in the Hamiltonian that are not $O(N)$ invariant. A typical example is given by the Euclidean action

$$S = \int d^d x \left[\sum_{i=1}^{N} \left(\frac{1}{2} |\nabla \phi_i|^2 + t_0 \phi_i^2 \right) + u_0 \left(\sum_{i=1}^{N} \phi_i^2 \right)^2 + v_0 \sum_{i=1}^{N} \phi_i^4 \right] , \tag{1}$$

where t_0, u_0, and v_0 are the dimensionful coupling constants related to the dimensionless couplings by $t = t_0 a^2$, $u = u_0 a^{4-d}$, and $v = v_0 a^{4-d}$. Here, a is the UV cutoff. Let us assume $u + v > 0$ in order to ensure that the action is bounded from below.

We are interested in studying this model in $d = 3$ when $N = 2$ (XY model) or $N = 3$ (Heisenberg model). This is of course very hard, so let us study this model in the $4 - \epsilon$ expansion (for any N).

a) Which term in the Euclidean action breaks the $O(N)$ symmetry?

b) In a series expansion at small u and v, the beta functions for the three coupling constants can be written as

$$\begin{aligned} -\beta_t &= \frac{dt}{d\ell} = c_1 t - 8(N+2)ut - 24vt + \cdots , \\ -\beta_u &= \frac{du}{d\ell} = c_2 u - 8(N+8)u^2 - 48uv + \cdots , \\ -\beta_v &= \frac{dv}{d\ell} = c_3 v - 96uv - 72v^2 + \cdots . \end{aligned} \tag{2}$$

What are the values of the numerical constants c_1, c_2, and c_3?

c) Find all the RG fixed points when $\epsilon \ll 1$. At each fixed point, calculate the scaling dimensions of the three operators that multiply t, u, and v, respectively, in the action (1). Note that there might be mixing between these operators.

d) Based on the values of the scaling dimensions you determined, which fixed point is the most stable one? Is there a critical value $N = N_c$ where the stability of the fixed points changes?

e) Sketch the RG flow diagram.

f) At the $O(N)$ invariant fixed point, do both operators multiplying u and v in the action (1) appear in the $\phi_i \times \phi_j$ OPE, or just a specific linear combination of them?

This question is interesting from the point of view of the numerical bootstrap. The simplest numerical bootstrap study of $O(N)$-invariant theories involves looking at a single correlator $\langle \phi_i \phi_j \phi_k \phi_\ell \rangle$. This correlator gives access only to operators that appear in the $\phi_i \times \phi_j$ OPE. So if an operator does not appear in this OPE, one might have to look into applying the conformal bootstrap to other 4-point functions.

Problem 2.

Problem Style: **PEN & PAPER**

Difficulty: **EASY**

Two-point functions of conserved currents in free theories.
Let us consider the Gaussian theory of N free massless real scalars ϕ_i, $i = 1, \ldots, N$, in d dimensions and Euclidean signature:

$$S_E = \frac{1}{2} \int d^d x \sum_{i=1}^{N} \partial_\mu \phi_i \, \partial^\mu \phi_i \, . \tag{3}$$

This CFT has a global $O(N)$ symmetry, and consequently a conserved current operator $J_{\mu ij}$, with J anti-symmetric in the i, j indices.

a) Determine the expression for the (canonically normalized) $J_{\mu ij}$ in terms of ϕ_i using the Noether procedure.

b) The two-point function of this current takes the form

$$\langle J_{\mu ij}(\vec{x}) J_{\nu kl}(0) \rangle = C \frac{\delta_{\mu\nu} - B \frac{x_\mu x_\nu}{x^2}}{|\vec{x}|^A} (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \, , \tag{4}$$

for some constants A, B, and C. (In Euclidean signature we do not make any distinction between upper and lower coordinate indices.) First determine the constants A and B from the requirement that $J_{\mu ij}$ is a conserved current and that it has the appropriate scaling dimension. These constants are independent of the details of the theory. (They are the same in any theory with an $O(N)$ conserved current operator.)

c) What is the two-point function (propagator) $\langle \phi_i(\vec{x}) \phi_j(\vec{y}) \rangle$? Be careful about the overall normalization.

d) Using the two-point function determined in part (c), determine the constant C appearing in eq. (4) for a theory of N free real scalars in d dimensions.

e) How would Eqs. (3) and (4) change in Lorentzian mostly plus signature?

f) In $d = 3$ one can consider another free CFT, namely the theory of N massless Majorana fermions. Let us choose a basis of real gamma matrices $\gamma^0 = i\sigma_2$, $\gamma^1 = \sigma_1$, $\gamma^2 = \sigma_3$, where σ_i are the Pauli matrices. With this gamma matrix convention, the Majorana fermions ψ_i are two-component real spinors. The Lorentzian action of the free theory is

$$S_L = -\frac{1}{2} \int d^3 x \sum_{i=1}^{N} \bar{\psi}_i \gamma^\mu \partial_\mu \psi_i \, , \tag{5}$$

where $\bar{\psi}_i = \psi_i^T (i\gamma_0)$.
Like the theory of free scalars, the theory of free fermions also has an $O(N)$ symmetry with a conserved current $J_{\mu ij}$. Repeat parts (a), (c), and (d) in this case. In particular, determine the expression for $J_{\mu ij}$ in terms of ψ_i, determine the fermion two-point function, and then calculate the current two-point function in order to extract C.